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Exact Solution of Poincaré Gauge Field Equations of Gravity with Torsion

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Under empty, static, and spherically symmetric conditions we find an exact metric solution of the Poincaré gauge field equations. The Schwarzschild metric solution is contained in the solution and we also obtain new gauge correction terms r^{-1} and $r^2 \ln r$.

1. INTRODUCTION

In recent years, the gauge theory of gravity has become a prime subject of research. Many authors have found a metric solution with double quality properties of a quadratic Poincaré gauge field theory (Baekler, 1981, 1983; Lenzen, 1985; Baekler *et al.*, 1988; Lee, 1983; Ramaswamy and Yasskin, 1979) proposed by Hehl *et al.* (1980). Under certain simplified conditions they obtained the metric gauge correction terms r^{-2} and r^2. Shao and Xu (1986) established the Poincaré gauge theory of gravity (PGTG) exactly by means of fiber bundle theory, in which there are two sets of field equations, the first the generalized Einstein field equations with gauge field, the second describing the relationship of spin currents and the geometrical properties of space time. We find an exact metric solution and torsion components from the first set of equations under empty, static, and spherically symmetric conditions. Our solution is composed of the Schwarzschild solution and three gauge correction terms r^{-1}, r^2, and $r^2 \ln r$, in which the terms r^{-1} and $r^2 \ln r$ are new gauge correction terms. The gauge correction terms have some interesting physical significance which we discuss.

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2. THE FIELD EQUATIONS OF PGTG

The Lagrangian of our field equations is

$$\mathscr{L} = C\mathscr{L}_m V + RV - \frac{\rho}{4} VF_{\mu\nu j}{}^{i} F^{\mu\nu j}{}_{i} - \frac{\rho'}{4} VQ^{k}{}_{\mu\nu} Q_k{}^{\mu\nu} \tag{1}$$

where $\mathscr{L}_m(\psi, \psi_{\nu\mu})$ is the matter field Lagrangian, which is zero in empty space-time, ρ, ρ' are gauge gravitational constants,

$$V = \det(V^i_\mu) = (-g)^{1/2}$$

$$C = 8\pi K$$

(K is the Newtonian gravitational constant) and

$$F_{\mu\nu}{}^{ij} = \partial_\mu B_\nu{}^{ij} + B_\mu{}^{i}{}_{k} B_\nu{}^{kj} - (\mu - \nu)$$

$$Q^{i}{}_{\mu\nu} = \partial_\mu V_\nu{}^{i} + B_\mu{}^{i}{}_{j} V_\mu{}^{j} - (\nu - \omega)$$

are the curvature and torsion tensors, and they are called gauge field strengths corresponding to the connection field $B_\mu{}^{j}{}_{i}$ and the vierbein field $V_\mu{}^{i}$.

From the Lagrangian (1) we get the first set of field equations of PGTG (Shao and Xu, 1986)

$$R_\mu{}^{i} - \frac{1}{2} V_\mu{}^{i} R = -CT_\mu{}^{i} - \rho t_\mu{}^{i} - \rho' \tau_\mu{}^{i} \tag{2}$$

where $T_\mu{}^{i}$ is the matter field energy-momentum tensor, which is zero in empty space-time; here the Lorentz indices are denoted by $i, j, k, \ldots = 0, 1, 2, 3$, and space-time indices are $\mu, \nu, \lambda, \ldots = 0, 1, 2, 3$.

Now we restrict ourselves to the condition of static spherical symmetry; then the metric can be put into the form

$$ds^2 = -e^{2\mu}\, dt^2 + e^{2\nu}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2$$

$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

Transform coordinates with $w^i = V_\mu{}^{i}\, dx^\mu$, where V^i_μ are the local Lorentz vierbein fields

$$V^0_\mu = (e^\mu, 0, 0, 0), \qquad V^1_\mu = (0, e^\nu, 0, 0)$$

$$V^2_\mu = (0, 0, r, 0), \qquad V^3_\mu = (0, 0, 0, r \sin\theta)$$

Thus in the new coordinate system the metric is

$$ds^2 = -(w^0)^2 + (w^1)^2 + (w^2)^2 + (w^3)^2 = \eta_{ij} w^i w^j$$

and we have $g_{\mu\nu} = V^i_\mu V^j_\nu \eta_{ij}$, $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$.

Then the field equations (2) in the coordinate system become

$$E_{ij} = -(CT_{ij} + \rho t_{ij} + \rho' \tau_{ij}) \tag{3}$$

where

$$E_{ij} = R_{ij} - \frac{1}{2}\eta_{ij}R$$

represent the Einstein tensors, and

$$t_{ij} = -\left(\frac{1}{2}F_i{}^n{}_{lm}F_{jn}{}^{lm}\right) + \frac{1}{8}(F_k{}^n{}_{lm}F^{klm}{}_n)\eta_{ij} \tag{4}$$

$$\tau_{ij} = -\frac{1}{2}Q_l{}^m{}_iQ^l{}_{jm} + \frac{1}{8}Q_l{}^{mn}Q_m{}^l{}_n\eta_{ij} \tag{5}$$

are the energy-momentum tensors of the connection and vierbein gauge fields and $\epsilon = 1 - \rho'$ is a new gauge gravitational constant.

3. THE SOLUTION OF THE FIELD EQUATIONS

Under the static, spherically symmetric condition, for the nonvanishing independent components of torsion the following assumption is chosen (Baekler, 1981, 1983; Lenzen, 1985; Baekler *et al.*, 1988; Lee, 1983; Ramaswamy and Yasskin, 1979):

$$\begin{aligned} Q^0{}_{01} &= f(r), & Q^2{}_{20} &= Q^3{}_{30} = k(r) \\ Q^1{}_{10} &= h(r), & Q^2{}_{21} &= Q^3{}_{31} = -g(r) \end{aligned} \tag{6}$$

On the fiber bundle, using the first set of Cartan structure equations

$$2B_i{}^k{}_j = -\theta_i{}^k{}_j + Q_i{}^k{}_j \tag{7}$$

we have

$$dw^i = \theta^i{}_{jk}w^j \wedge w^k$$

so the nonvanishing independent $\theta^i{}_{jk}$ can be calculated:

$$\begin{aligned} \theta^0{}_{10} &= \mu' e^{-\nu}, & \theta^2{}_{12} &= \frac{1}{r}e^{-\nu} \\ \theta^3{}_{13} &= \frac{1}{r}e^{-\nu}, & \theta^3{}_{23} &= \frac{1}{r}\operatorname{ctg}\theta \end{aligned} \tag{8}$$

Put (6) and (8) into (7); the nonvanishing coefficients of connection are

$$B^0{}_{10} = -\mu' e^{-\nu} - f, \qquad B^1{}_{10} = h, \qquad B^2{}_{20} = B^3{}_{30} = k$$

$$B^2{}_{21} = B^3{}_{31} = \frac{1}{r}e^{-\nu} - g, \qquad B^3{}_{23} = -\frac{1}{r}\operatorname{ctg}\theta$$

Then on the fiber bundle the nonvanishing independent connection 1-form

$$w^i_j = B^i{}_{kj} w^k$$

may be obtained:

$$\begin{aligned}
w^0_1 &= (\mu' e^{-\nu} + f) w^0 + h w^1, & w^2_1 &= \left(\frac{1}{r} e^{-\nu} - g\right) w^2 \\
w^3_1 &= \left(\frac{1}{r} e^{-\nu} - g\right) w^3, & w^3_2 &= \frac{1}{r} \operatorname{ctg} \theta w^3 \\
w^2_0 &= k w^2, & w^3_0 &= k w^3
\end{aligned} \tag{9}$$

From the second set of Cartan structure equations

$$\Omega^i_j = dw^i_j + w^i_k \wedge w^k_j = R^i{}_{jkl} w^k \wedge w^l$$

where $R^i{}_{jkl}$ are the curvature coefficients on the fiber bundle, using expressions (9) and making a suitable choice of bases, we have the following nonvanishing independent curvature components:

$$\begin{aligned}
R^0{}_{110} &= (\mu' e^{\mu-\nu} + e^{\mu} f)' \, e^{-\mu-\nu} \equiv A \\
R^2{}_{323} &= \frac{1}{r^2} + k^2 - \left(\frac{1}{r} e^{-\nu} - g\right) \equiv L \\
R^0{}_{202} &= R^0{}_{303} = -(f + \mu' \, e^{-\nu}) \left(\frac{e^{-\nu}}{r} - g\right) \equiv J \\
R^1{}_{212} &= R^1{}_{313} = \frac{1}{r} e^{-\nu} (rg - e^{-\nu})' + hk \equiv H \\
R^1{}_{202} &= R^1{}_{303} = (\mu' \, e^{-\nu} + f) k \equiv G \\
R^0{}_{212} &= R^0{}_{313} = (rk)' \, \frac{e^{-\nu}}{r} - h\left(\frac{1}{r} e^{-\nu} - g\right) \equiv -D
\end{aligned} \tag{10}$$

Substituting (10) into (3), we find the following nonvanishing Einstein tensors:

$$\begin{aligned}
E_{00} &= -2H - L \\
E_{01} &= 2D \\
E_{10} &= 2G \\
E_{11} &= L - 2J \\
E_{22} &= E_{33} = H + A - J
\end{aligned}$$

Using (10) and (6) in (4) and (5), respectively, we get

$$t_{00} = A^2 + 2J^2 - 2G^2 + 2D^2 - 2H^2 - L^2$$

$$t_{01} = -(4DJ - 4HG)$$

$$t_{10} = -(4DJ - 4HG)$$

$$t_{11} = -A^2 + 2D^2 + 2J^2 - 2H^2 - 2G^2 + L^2$$

$$t_{22} = t_{33} = A^2 - L^2$$

$$\tau_{00} = \frac{1}{2}f^2 - \frac{1}{2}h^2 - k^2 - g^2$$

$$\tau_{01} = 2kg$$

$$\tau_{10} = 2kg$$

$$\tau_{11} = -k^2 - g^2$$

$$\tau_{22} = \tau_{33} = \frac{1}{2}f^2 - \frac{1}{2}h^2$$

Substituting these into the field equations (3), we obtain the following independent equations:

$$-2H - L + \rho(A^2 + 2J^2 - 2G^2 + 2D^2 - 2H^2 - L^2) + \rho'\left(\frac{1}{2}f^2 - \frac{1}{2}h^2 - k^2 - g^2\right) = 0$$

$$2D - \rho(4DJ - 4HG) + 2\rho' kg = 0$$

$$2G - \rho(4DJ - 4HG) + 2\rho' kg = 0$$

$$-2J + L + \rho(-A^2 + 2J^2 - 2G^2 + 2D^2 - 2H^2 + L^2) - \rho'(k^2 + g^2) = 0$$

$$-J + H + A + \rho(A^2 - L^2) + \rho'\left(\frac{1}{2}f^2 - \frac{1}{2}h^2\right) = 0$$

In order to find the solution, we assume that

$$f = -h, \qquad g = k, \qquad e^{2\mu} = e^{-2\nu} \quad (\text{i.e., } \mu + \nu = 0)$$

Then we obtain three nontrivial independent equations

$$D = G, \qquad A + L = -\frac{1}{2\rho}, \qquad G = -\frac{2}{3}\rho' k^2 \tag{11}$$

Replacing (11) by (10), then

$$\frac{k}{r}e^{\mu}+k'\,e^{\mu}+hk-\frac{h}{r}e^{\mu}=-k\,e^{\mu'}+kh$$

$$\mu''\,e^{2\mu}+2\mu'^2\,e^{2\mu}-(he^{\mu})'+\frac{1}{r^2}+\frac{2k}{r}e^{\mu}-\frac{e^{\mu}}{r^2}=-\frac{1}{2\rho}$$

$$\mu'\,e^{\mu}-h=-\frac{2}{3}\rho'k$$

and with the assumption $x=k\,e^{\mu}$, $y=h\,e^{\mu}$, and $z=e^{2\mu}$, we obtain the final simplified equations:

$$x'+\frac{x}{r}-\frac{y}{r}=0$$

$$z''-2y'+\frac{4x}{r}-\frac{2z}{r^2}+\frac{2}{r^2}=-\frac{1}{\rho}$$

$$z'-2y+\frac{4}{3}\rho'x=0$$

The solutions are

$$x=\left(c_1-\frac{1}{6\rho\rho'}\right)r+\frac{r\ln r}{2\rho\rho'}-\frac{c_2}{3r^2}$$

$$y=\left(2c_1+\frac{1}{6\rho\rho'}\right)r+\frac{r\ln r}{\rho\rho'}+\frac{c_2}{3r^2}$$

$$z=1+\left(2c_1+\frac{5}{18\rho}-\frac{1}{3\rho\rho'}-\frac{2}{3}c_1\rho'\right)r^2-\left(\frac{2}{3}c_2+\frac{4}{9}\rho'c_2\right)\frac{1}{r}+\left(\frac{1}{\rho\rho'}-\frac{1}{3\rho}\right)r^2\ln r$$

where c_1 and c_2 are integral constants. For physical reasons we choose $c_1=0$ and $c_2=3km$ (m is the mass of gravitational matter).

Then the nonvanishing components of metric and torsion are

$$-g_{00}=\left[1-\frac{2km}{r}-\frac{4km\rho'}{3}\frac{1}{r}+\left(\frac{5}{18\rho}-\frac{1}{3\rho\rho'}\right)r^2+\left(-\frac{1}{3\rho}+\frac{1}{\rho\rho'}\right)r^2\ln r\right]$$

$$g_{11}=(-g_{00})^{-1}=\left[1-\frac{2km}{r}-\frac{4km\rho'}{3}\frac{1}{r}+\left(\frac{5}{18\rho}-\frac{1}{3\rho\rho'}\right)r^2+\left(-\frac{1}{3\rho}+\frac{1}{\rho\rho'}\right)r^2\ln r\right]^{-1}$$

$$g_{22}=r^2$$

$$g_{33} = r^2 \sin^2 \theta$$

$$Q^0{}_{01} = Q^1{}_{01} = \left[\frac{1}{6\rho\rho'} r + \frac{r \ln r}{\rho\rho'} + \frac{km}{r^2}\right](-g_{00})^{-1/2}$$

$$Q^2{}_{20} = Q^3{}_{30} = Q^2{}_{12} = Q^3{}_{13}$$

$$= \left[-\frac{1}{6\rho\rho'} r + \frac{r \ln r}{2\rho\rho'} - \frac{km}{r^2}\right](-g_{00})^{-1/2}$$

4. DISCUSSION

This solution is composed of Schwarzschild metric terms and gauge correction terms r^{-1}, r^2, and $r^2 \ln r$ in the matrix of the metric $g_{\mu\nu}$.

The terms $1/r$ is a new, additional Newtonian potential mediated by the vierbein fields V^i_μ; the potential describes the gravitational behavior of macroscopic matter and corrects the metric in the long range of space-time.

The terms r^2 and $r^2 \ln r$ are both confinement potentials mediated by the connection coefficient fields $B_\mu{}^{ij}$; the confinement potentials play roles in the strong interaction and relate to the metric in the short range of space-time. The property of confinement potentials, the gauge correct term r^2, has been applied in strong gravity and the strong interaction.

Because the connection coefficient fields $B_\mu{}^{ij}$ may be used to mediate the strong interaction, they offer a possible way to unify the gravitational interaction and the strong interaction at some level.

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